MATH 211 EXAM II
Fall 2005
November 9

Name: KEY

Instructions. Please write your name when you first receive this test, write clearly and justify all your answers. Every question or part of a question is worth two points (a total of 24 points). This test sheet is to be submitted with your answers.

(1) Prove that $n!$ is $O(n^n)$. Show carefully the witnesses C and k in your proof.
(2) Prove that $1^3 + 2^3 + \cdots + n^3$ is $O(n^4)$. Show carefully the witnesses C and k in your proof.
(3) Find the remainder of 5554781334323
  (a) when divided by 5
  (b) when divided by 9
(4) (a) Write the decimal integer 1021 in base 3.
  (b) represent the number $(4B1AF)_{16}$ in decimal. (make sure you provide the final answer.)
(5) Using just 7-cent and 10-cent stamps, any amount of money (or postage) greater than or equal to k can be made. Find k and prove the statement using mathematical induction.
(6) (a) Use the Euclidean algorithm to find AND represent $gcd(18, 13)$ as a linear combination of 18 and 13.
  (b) Find an inverse of 13 mod 18
  (c) Find an inverse of 18 mod 13
  (d) Solve the system of equations $\begin{cases} x \equiv 7 & \mod 13 \\ x \equiv 3 & \mod 18 \end{cases}$
  (make sure you write down the final answer for the solution(s) x)
(7) Use mathematical induction to show that $n^2 - 7n + 12$ is a nonnegative integer whenever $n \geq 3$.

Do one or more of these **Bonus Problems.** You receive two points for each 'round' answer:

B1 *Déja Vue.* Use induction to show that a set with n elements has exactly 2^n subsets.
B2 *moderately challenging.* Write a pseudo-code for an $O(n)$ function that reports the second largest value of a list with n integers. (Note: you still get one point if your function is $O(n^2)$.)
B3 *Mind TeaZer.* Use Induction to show that given any subset A with $n + 1$ elements of the set$\{1, 2, 3, \cdots, 2n\}$, then there is an integer in A that divides another integer in A.

Bon Chance

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(1) $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$
but each term above (in the right hand side) is $\leq n$,
since we have n terms, we get $n! < \underbrace{n \cdots n} = n^n$
This is true for all $n \geq 1$ so $n! \leq C \cdot n^n \ \forall n \geq 1$
so $K = C = 1$.

2) Similar to (1): $\left.\begin{array}{l} 1^3 \leq n^3; \\ 2^3 \leq n^3 \\ \vdots \\ n^3 \leq n^3 \end{array}\right\}$ $1^3 + 2^3 + \cdots + n^3 \leq n \cdot n^3 = n^4$
so $C = K = 1$.

3 a) 3 (because a number mod 5 is the first digit

b) 8 [a number mod 9 equal the sum of its digits mod 9]

4) Divide repeately by 3 and "record" the remainders:

a)

$1021 = 3 \times 340 + 1$

$340 = 3 \times 113 + 1$

$113 = 3 \times 37 + 2$

$37 = 3 \times 12 + 1$

$12 = 3 \times 4 + 0$

$4 = 3 \times 1 + 1$

$1 = 3 \times 0 + 1$

So write the remainder starting with the last one first:

$(1101211)_3 = (1021)_{10}$

b) $(4B1AF)_{16} = 15 + 10 \times 16 + 1 \times 16^2 + 11 \times 16^3 + 4 \times 16^4$

$= 15 + 160 + 256 + 45{,}056 + 262{,}144 = 307{,}631.$

5) K is 54 (note that 47, 48, 49, 50, 51, 52 all work, but not 53!)

<u>Base case</u>: $k = 54$: 2 sevens & 4 tens.

Ind. Hyp.: suppose it works for <u>some</u> $k \geq 54$.

Ind. Step: we need to show that it works for $k+1$:

- suppose we have 2 tens in the representation of k, then replace them by three seven-cent stamps. Hence in this case $(k+1)$ can also "represented" by sevens & tens.
- we need to be able to represent $(k+1)$ in all cases, so suppose the representation of k does not have 2 tens, i.e, it has one or less ten-cent stamps:
  - i) if it has no tens, then it must have at least 7 sevens (since $k \geq 54$)
  - ii) if it has one ten, remove it, so the remaining amount is <u>at least</u> 44, without tens, it still must have 7 sevens! (or more)

  in either case, replace 7 sevens by 5 tens, making $(k+1)$.

It follows by the principle of mathematical induction that any integer amount ≥ 54 can be made using sevens & tens.

6) $18 = 13 \times 1 + 5$

$13 = 5 \times 2 + 3$

(a) $5 = 3 \times 1 + 2$

$3 = 2 \times 1 + 1 \leftarrow$ gcd

$2 = 1 \times 2 + 0$

now going backwards:

$$
\begin{aligned}
1 &= 3 - 2 \times 1 \\
&= 3 - (5 - 3) = \underline{3} \times 2 - \underline{5} \\
&= (\underline{13} - \underline{5} \times 2) \times 2 - 5 \\
&= \underline{13} \times 2 - \underline{5} \times 4 - \underline{5} = 13 \times 2 - \underline{5} \times 5 \\
&= 13 \times 2 - (\underline{18} - \underline{13}) \times 5 \\
&= 13 \times 2 - \underline{18} \times 5 + \underline{13} \times 5
\end{aligned}
$$

so $1 = 13 \times 7 - \underline{18} \times 5$

(verify: $13 \times 7 - 18 \times 5 = 91 - 90 = 1$)

(b) From part (a) we see that $\underline{13} \times 7 - \underline{18} \times 5 \equiv 1 \mod 18$

hence $13 \times 7 \equiv 1 \mod 18$, hence 7 is an inverse of 13 mod 18

(c) similarly, (-5) is an inverse of 18 mod 13.

(d) we use the chinese remainder theorem:

since 13 & 18 are relatively prime we know that there is a solution to this system:

$$
\left.\begin{aligned} m_1 &= 13 \\ m_2 &= 18 \end{aligned}\right\} \Rightarrow
\left.\begin{aligned} M_1 &= 13 \times 18/13 = 18 \\ M_2 &= 13 \times 18/18 = 13 \end{aligned}\right\} \Rightarrow
\begin{aligned} &\overline{M}_1 \mod 13 = -5 \ [\text{part (c)}] \\ &\overline{M}_2 \mod 18 = 7 \ [\text{part (b)}] \end{aligned}
$$

so the solution is $-5 \times 18 \times 7 + 7 \times 13 \times 3 = -357$

(Note: adding a multiple of 13×18 gives another solution, e.g. $-357 + 2 \times 13 \times 18 = 111$)

(7) Base case: $n = 3$: $3^2 - 7 \times 3 + 12 = 0 \geq 0$.

Ind. hyp.: suppose that for <u>some</u> $k \geq 3$, $k^2 - 7k + 12 \geq 0$

Ind. step: we want to show that $(k+1)^2 - 7(k+1) + 12 \geq 0$.

$(k+1)^2 - 7(k+1) + 12 = k^2 + 2k + 1 - 7k - 7 + 12 = (k^2 - 7k + 12) + (2k - 6)$

But $(k^2 - 7k + 12) \geq 0$ by hypothesis; & $2k \geq 6$ since $k \geq 3$ so that $(k+1)^2 - 7(k+1) + 12$ is the sum of two non-negative terms, hence it is non-negative. The result follows by math. ind.